Connection b/w homotopy & cohomology:

Thm: X CW-complex $\Rightarrow [X, S^1] \simeq H^1(X, \mathbb{Z})$.

Recall: $C_*(X)$ cellular chain complex (C_k = free $\mathbb{Z}$-module gen^d by the k-cells)
coeff^t $\langle \partial e, e' \rangle$ of $(k-1)$ cell e' in ∂e is degree of attaching map of e onto e'.
$C^*(X, G) = \mathrm{Hom}(C_*(X), G)$ with dual differential δ.

Pf: α generator of $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$; define $\Phi : [X, S^1] \longrightarrow H^1(X, \mathbb{Z})$
(ie: 1-cell of $S^1 \underset{\alpha}{\mapsto} 1$) $\quad [f : X \to S^1] \longmapsto f^*\alpha$

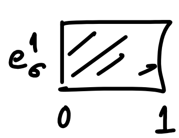
- Φ surjective: let $\xi \in C^1(X, \mathbb{Z})$ with $\delta\xi = 0$. Want $f : X \to S^1$ st. $f^*\alpha = [\xi]$.
  - → f on 0-cells $\longmapsto$ base point $p \in S^1$
  - → f on 1-cell e^1_σ: so that $f_{|e^1_\sigma} : I \to S^1$ winds $\xi(e^1_\sigma)$ times around S^1
  - → consider a 2-cell $e : D^2 \to X$: then $\delta\xi = 0 \Rightarrow \xi(\partial e) = \sum_\sigma \langle \partial e, e^1_\sigma \rangle\, \xi(e^1_\sigma) = 0$
    Hence winding number of f along boundary loop of e is zero.
    ie. represents $0 \in \pi_1(S^1)$; so f extends over the 2-cell e.
  - → then by induction, assume f defined over k-skeleton ($k \geq 2$),
    let e be a $(k+1)$-cell w/ attaching map $\phi : S^k \to X$, then $[f \circ \phi]$
    trivial in $\pi_k(S^1) = 0$, so f extends over e .... → get f everywhere.

- Φ injective: $f_0, f_1 : X \to S^1$ st. $f_0^*\alpha = f_1^*\alpha$, want to show f_0, f_1 homotopic.
  Can assume f_0, f_1 cellular ie. map X^0 to base pt $p \in S^1$.
  Then $f_i^*\alpha$ is rep^d by cellular cochain $\beta_i : \underset{\text{1-cell in } X}{e^1_\sigma} \longmapsto$ winding number of $f_{i|e^1_\sigma}$ around S^1.

  $f_0^*\alpha = f_1^*\alpha \Rightarrow \exists$ 0-cochain $\eta \in C^0(X, \mathbb{Z})$ st. $\delta\eta = \beta_1 - \beta_0$
  ie. if $e^1_\sigma : I \to X$ has vertices $e^1_\sigma(0), e^1_\sigma(1)$, $\beta_1(e^1_\sigma) - \beta_0(e^1_\sigma) = \eta(e^1_\sigma(1)) - \eta(e^1_\sigma(0))$ (*)

  Now: $f_0 \sqcup f_1 : X \times \{0,1\} \to S^1$, want to extend to $F : X \times [0,1] \to S^1$.

  → on $X^0 \times [0,1]$: choose st. if $x \in X^0$, $F_{|\{x\} \times [0,1]}$ winds $\eta(x)$ times around S^1
  → on $X^1 \times [0,1]$: boundary of $e^1_\sigma \times [0,1]$ consists of 4 arcs



$e^1_\sigma \times \{0\}$	where f_0 winds	$\beta_0(e^1_\sigma)$ around S^1
$e^1_\sigma \times \{1\}$	f_1	$\beta_1(e^1_\sigma)$
$e^1_\sigma(0) \times [0,1]$	F	$\eta(e^1_\sigma(0))$
$e^1_\sigma(1) \times [0,1]$	F	$\eta(e^1_\sigma(1))$

Equation (*) ensures total winding number along $\partial(e^1_\sigma \times I)$ is 0
so F extends over cell.

→ higher-dim. cells: induction, no obstruction since $\pi_k(S^1) = 0 \ \forall k > 1$.
$\Rightarrow$ get homotopy F b/w f_0 & f_1 ✓

▲

This generalizes to:

Thm. X CW-complex, G abelian group, $n \geq 1 \Rightarrow H^n(X;G) \simeq [X, K(G,n)]$.

Pf.: Let $K = K(G,n)$ (our favorite one). Start with a preferred element $\alpha \in H^n(K, G)$:
Note $H_n(K) \cong G$ by Hurewicz, so by univ. coeff. thm, $H^n(K,G) = \mathrm{Hom}(G,G) \ni \alpha = id$.
Recall $K(G,n)$ has n-cells for generators of G, attach $(n+1)$-cells along relations.
Given an n-cell e^n_σ corresponding to $g_\sigma \in G$ one of the generators,
define $\alpha(e^n_\sigma) = g_\sigma$. This gives $\alpha \in C^n(K,G)$.
And for an $(n+1)$-cell e^{n+1}, $\alpha(\partial e^{n+1}) = (\text{relation among the } g_\sigma) = 0$ in G.
so $\delta\alpha = 0$ ✓.

- Now repeat proof before. We map
$$[X,K] \longrightarrow H^n(X,G)$$
$$[f] \longmapsto f^*\alpha$$

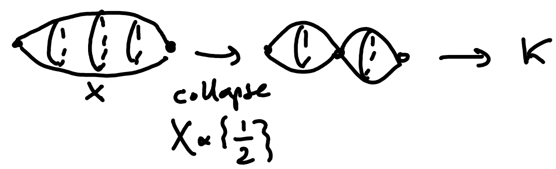
- Surjectivity: let $\xi \in C^n(X,G)$ with $\delta(\xi) = 0$. Want $f: X \to K$ s.t. $f^*\alpha = [\xi]$.
  - → f on 0-cells ... $(n-1)$-cells $\mapsto$ base point $x_0 \in K$
  - → on an n-cell e^n_σ: choose it so that $f_{|e^n_\sigma}: (I^n, \partial I^n) \to (K, x_0)$ represents the element $\xi(e^n_\sigma)$ of $\pi_n(K) = G$.
  - → on an $(n+1)$-cell e^{n+1} with attaching map $\phi: S^n \to X$:
  $$\delta\xi = 0 \Rightarrow \xi(\partial e^{n+1}) = \sum_\sigma \langle \partial e^{n+1}, e^n_\sigma \rangle \, \xi(e^n_\sigma) = 0 \in G$$
  so $f \circ \phi$ is trivial in $\pi_n(K) = G$, hence f extends over the $(n+1)$-cell.
  - → then by induction on higher dim. cells of X: if f defined on k-skeleton $(k \geq n+1)$ then for a $(k+1)$ cell w/ attaching map ϕ, $[f \circ \phi] = 0$ in $\pi_k(K) = 0$ so f extends.

- Injectivity: adapt previous proof likewise.
given cellular $f_0, f_1: X \to K$ s.t. $f_0^*\alpha = f_1^*\alpha$,
represent $f_i^*\alpha$ by cellular cochain $\beta_i: e^n_\sigma \mapsto$ class of $f_{i|e^n_\sigma}$ in $\pi_n(K, x_0) = G$.
$\exists$ $(n-1)$-cochain $\eta \in C^{n-1}(X, \mathbb{Z})$ s.t. $\delta\eta = \beta_1 - \beta_0$. Use η to build homotopy b/w f_0 & f_1 on the $(n-1)$-skeleton of X ($F_{|e^{n-1} \times I}$ represents $\eta(e^{n-1}) \in \pi_n(K, x_0)$)
Then the equation $\delta\eta = \beta_1 - \beta_0$ guarantees we can extend F over $(n\text{-cells}) \times I$ and no obstruction to extending over higher dim. cells since $\pi_{\geq n+1}(K) = 0$. ∎

Rmk: same results hold with base pt preserving maps, $H^n(X,G) \simeq \langle X, K(G,n)\rangle$.
(connectedness of K & cellular approximation ensure the two are the same).

• Question: how come $[X,K]$ or $\langle X,K\rangle$ has a group structure ??
(for which X, K is this the case?)

1) Observe: $\langle S^n, K\rangle = \pi_n(K)$ has group structure.
More generally this holds for suspensions – $\forall X, K$, $\langle SX, K\rangle$ is a group
namely, $f, g : SX \to K$ base pt preserving $\Rightarrow$ $f+g : SX \longrightarrow SX \vee SX \xrightarrow[f\vee g]{} K$

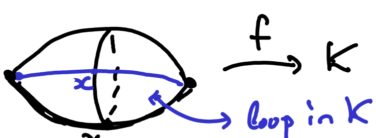


There's a small issue w/ base pts, want all of $\{x_0\}\times I$ to be base pt for this to really work; so instead consider $\Sigma X = SX/\{x_0\}\times I$ ($\underset{he}{\simeq} SX$, so no big deal...)
reduced suspension

2) Define the loop space $\Omega K = \{ f : (S^1, s_0) \to (K, x_0)\}$ with compact-open topology
(basis: maps st. f(given compact subset) $\subseteq$ (given open subset))
(if domain compact & target = metric space, $\Leftrightarrow$ unif. convergence)

Then we have the adjoint relation $\underline{\underline{\langle \Sigma X, K\rangle = \langle X, \Omega K\rangle}}$.

Namely, $f : \Sigma X \to K \iff g : X \to \Omega K$
$x \mapsto f_{|\{x\}\times I}$



(note $x_0 \mapsto f_{|\{x_0\}\times I}$ = const. loop at base pt of K)

Note $\langle X, \Omega K\rangle$ has a natural group structure coming from composition of loops!
$f+g : X \xrightarrow{f\times g} \Omega K \times \Omega K \xrightarrow{\text{composition}} \Omega K$

Under adjoint relation this is the same group law as for $\langle \Sigma X, K\rangle$.

★ Taking $X = S^{n-1}$ in adjoint relation $\Rightarrow$ $\underline{\pi_n(K) = \pi_{n-1}(\Omega K)}$ (already mentioned in 1st lecture)
(vs. remember $\tilde{H}_n(X) = \tilde{H}_{n+1}(\Sigma X)$).

- in particular for $K = K(G,n)$, $\Omega K(G,n)$ has $\pi_{n-1} = G$, others zero.
  In fact $K(G,n-1)$ is a CW-approximation to $\Omega K(G,n)$ (weakly h.e.)

- Another perspective: the <u>path-loop fibration</u>

  $PX = \{ f : I \to X \,/\, f(0) = x_0 \}$ with compact-open topology

  - PX is contractible $\left(\text{retract to constant path by considering } f \overset{\varphi_s}{\longmapsto} (t \mapsto f(st)) \quad \varphi_1 = id,\ \varphi_0 = \text{const.} \right)$

  - the map $\begin{array}{ccc} PX & \xrightarrow{p} & X \\ f & \longmapsto & f(1) \end{array}$ is a fibration, with fiber $p^{-1}(x_0) = \Omega X$.

  Hence l.e.s. gives again $\pi_n(X) \xrightarrow{\sim} \pi_{n-1}(\Omega X)$.

In fact: <u>Prop:</u> $F \to E \xrightarrow{p} B$ fibration or fiber bundle with E contractible
$\Rightarrow \exists$ weak homotopy equivalence $F \simeq \Omega B$.

<u>Pf:</u> Let $f_t : E \to E$ homotopy $f_0 =$ const map to x_0 to $f_1 = id$.

Define a map $E \xrightarrow{\varphi} PB$

$x \longmapsto (t \mapsto p(f_t(x)))$

map $I \to B$ starting at $p(x_0) = b_0 \in B$
ending at $p(x)$.

Then
$$\begin{tikzcd} F := p^{-1}(b_0) \arrow[r, "\text{incl.}"] \arrow[d, "\varphi"'] & E \arrow[r, "p"] \arrow[d, "\varphi"] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, equal] \\ \Omega B \arrow[r, "\text{incl.}"'] & PB \arrow[r] & B \end{tikzcd}$$
commutative diagram; compare l.e.s. for the two fibrations

isoms on $\pi_n = 0\ \forall n$ (contractible

isoms on $\pi_n\ \forall n$

$\Rightarrow F \xrightarrow{\varphi} \Omega B$ weak h.e. ◀

So e.g. $S^1 \to S^\infty \to \mathbb{CP}^\infty$ gives weak h.e. $S^1 \to \Omega \mathbb{CP}^\infty$
and similarly $U(n) \to \Omega G_n(\mathbb{C}^\infty)$ (+ similarly for $O(n)$, $Sp(n)$).